## 5.45 GAUSS'S THEOREM OF DIVERGENCE

**(Relation between surface integral and volume integral)**

**Statement.** The surface integral of the normal component of a vector function $F$ taken around a closed surface $S$ is equal to the integral of the divergence of $F$ taken over the volume $V$ enclosed by the surface $S$.

Mathematically

$$\iint_S F \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \overline{F}\, dv$$

**Proof.** Let $\overline{F} = F_1 i + F_2 j + F_3 k$.

Putting the value of $F$, $n$ in the statement of the divergence theorem we have

$$\iint_S (F_1 i + F_2 j + F_3 k)\, \hat{n}\, ds = \iiint_V \left( i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)(F_1\, i + F_2 j + F_3 k)\, dx\, dy\, dz$$

$$= \iiint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz \quad ...(1)$$

We require to prove (1).

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz$.

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[ \int_{z = f_1(x, y)}^{z = f_2(x, y)} \frac{\partial F_3}{\partial z}\, dz \right] dx\, dy$$

$$= \iint_R \Big[ F_3(x, y, z) \Big]_{z = f_1(x, y)}^{z = f_2(x, y)} dx\, dy$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]\, dx\, dy \quad ...(2)$$

For the upper part of the surface *i.e.* $S_2$, we have

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$$dx\,dy = ds_2 \cos r_2 = n_2 \cdot k\, ds_2$$

Again for the lower part of the surface *i.e.* $S_1$ we have,

$$dx\,dy = -\cos r_1, ds_1 = \hat{n}_1 \cdot k\, ds_1$$

$$\iint_R F_3(x, y, f_2)\, dx\, dy = \iint_{S_2} F_3\, \hat{n}_2 \cdot k\, ds_2$$

and $$\iint_R F_3(x, y, f_1)\, dx\, dy = -\iint_{S_1} F_3\, \hat{n}_1 \cdot k\, ds_1$$

Putting these values in (2) we have

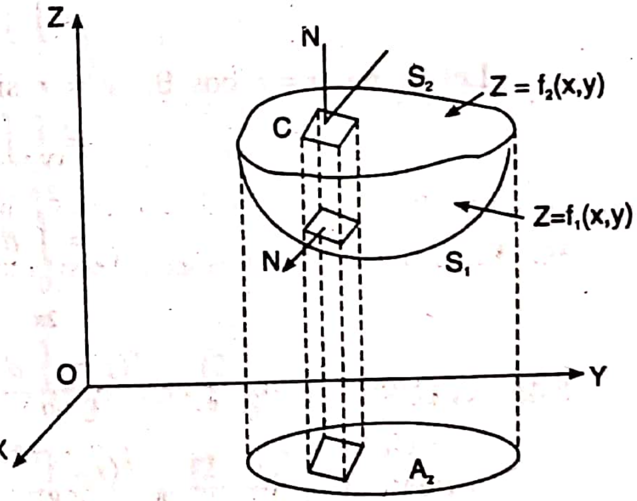
$$\iiint_V \frac{\partial F_3}{\partial z}\, dv = \iint_{S_2} F_3\, \hat{n}_2 \cdot k\, ds_2 + \iint_{S_1} F_3\, \hat{n}_1 \cdot k \cdot ds_1$$

$$= \iint_S F_3 \hat{n} \cdot k\, ds \qquad \text{...(3)}$$

Similarly, it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\, dv = \iint_S F_2\, \hat{n} \cdot j\, ds \qquad \text{...(4)}$$

$$\iiint_V \frac{\partial F_1}{\partial x}\, dv = \iint_S F_1\, \hat{n} \cdot i\, ds \qquad \text{...(5)}$$




Adding (3), (4) & (5) we have

$$\iiint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dv$$

$$= \iint_S (F_1 i + F_2 j + F_3 k) \cdot \hat{n} \cdot ds \qquad \text{...(4)}$$

or $$\iiint_V (\nabla \cdot \overline{F})\, dv = \iint_S \overline{F} \cdot \hat{n}\, ds$$

**Proved**